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Heterotic models without fivebranes

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Abstract

After discussing some general problems for heterotic compactifications involving fivebranes we construct bundles, built as extensions, over an elliptically fibered Calabi–Yau threefold. For these we show that it is possible to satisfy the anomaly cancellation topologically without any fivebranes. The search for a specific Standard Model or GUT gauge group motivates the choice of an Enriques surface or certain other surfaces as base manifold. The burden of this construction is to show the stability of these bundles. Here we give an outline of the construction and its physical relevance. The mathematical details, in particular the proof that the bundles are stable in a specific region of the Kähler cone, are given in the mathematical companion paper math.AG/0611762.

Keywords: Stable vector bundles; Calabi–Yau threefolds; Standard Model; Heterotic string

1. On problems caused by fivebranes and the H-field

Let us recall first some problems with fivebranes and the H-field which, although known in principle, have sometimes been neglected in the literature. These problems lead to some difficulties in the usage of bundles V (say in the observable sector) over a Calabi–Yau space X for a heterotic string compactification. Apart from the standard embedding $F = R$ the anomaly cancellation condition leads in general (already when read just on a cohomological level) to the occurrence of a fivebrane class W. We argue that this enforces the existence of a non-vanishing H-field which is of a markedly singular character. Even when it is possible to avoid this (for example by interpreting this class W as the $c_2(V_{\text{hid}})$ of a hidden bundle) one encounters still the same problematic occurrence of the H-field: this is when one realizes that the anomaly condition has actually to be solved on the form level already. A non-trivial H-field however is known to lead, via supersymmetry, to compatibility conditions on the underlying space geometry X, which turns out to be non-Kähler. This has the consequence that the usual Donaldson–Uhlenbeck–Yau (DUY) theorem is no longer applicable. This theorem assured the solvability of the equation of motion $F^{a\bar{b}} J_{a\bar{b}} = \frac{1}{2} F \wedge J^2 = 0$ on the form level from the condition $c_1(V) J^2 = 0$ on the cohomological level for bundles V stable with respect to a suitable Kähler class J. A generalization for the non-Kähler case [2] gives the solvability of the equation of motion

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for X having a Gauduchon metric, using an appropriate stability notion. Still one has to solve the non-linear anomaly equation exactly in a manner which goes beyond the perturbative arguments arguing by corrections order by order [1]. Recently first steps in that direction were made [26].

1.1. Problems when fivebranes are present

Besides solving the equation of motion $F \wedge J^2 = 0$, the field strength F must also obey the anomaly equation

$$dH = \frac{\alpha'}{4}(\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F)\,. \tag{1.1}$$

Many known vector bundle constructions, such as the spectral cover method, explicitly violate the ensuing topological condition $c_2(X) = c_2(V)$. So one needs to introduce a number of space–time filling fivebranes and has to use the generalized anomaly equation

$$dH = \frac{\alpha'}{4}\left(\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F + 16\pi^2 \delta_W\right). \tag{1.2}$$

Here a further 4-form arises which is supported on the codimension four world-volume of W; this is a 4-form with distribution coefficients, i.e. a current. The possibility of having a $W \neq 0$ was first treated on a computational level in [3].

The necessary condition for solving (1.2) is the corresponding integrability condition

$$c_2(X) = c_2(V) + W \tag{1.3}$$

(assuming $c_1(V) = 0$) where W has to be realizable by an effective class of holomorphic curves. However, the condition actually to be solved is (1.2) on the form level.

The presence of fivebranes prevents heterotic models from being interpreted as perturbative non-linear sigma models. Studies of stability of $(0, 2)$ models usually assume the absence of fivebranes. Especially problematic is that the delta function δ_W is not cancelled by another term on the right hand side which indicates that one cannot have a consistent solution without an H-field

$$W = c_2(X) - c_2(V) \neq 0 \Longrightarrow H \neq 0. \tag{1.4}$$

The non-trivial H-field must fulfil even $dH \neq 0$. More precisely, H must be chosen such that dH contains the relevant current. The singular character of H becomes especially severe when one realizes that in (1.2) a consistent H-solution occurs simultaneously on the right hand side, inside the connection with torsion $\omega + aH$ from which the tr $R \wedge R$-term is computed (there is an important issue in this framework as to which a has to be used in different places; for our purposes we do not need to enter this discussion).

One solution of this singularity problem, caused by the current contribution, could lie in dissolving the delta-function-like fundamental brane (given by the small instanton) into a smooth solitonic (gauge-)field configuration. However, although such fivebrane contributions W can occur as singular limits of gauge bundles, where (a part of) the curvature term tr $F \wedge F$ degenerates to a delta-function source, we are not concerned here with such small instanton transitions. These change the rank (V) and/or $c_3(V)$ [3,8,9]; so if one insists on certain values for these invariants of $V_{(\mathrm{obs})}$ (say from phenomenological investigations in the observable sector) giving a $W \neq 0$, it is of no help to change to another bundle $\tilde{V}_{(\mathrm{obs})}$ where these phenomenologically decisive invariants differ and just W is absorbed into $c_2(\tilde{V}_{(\mathrm{obs})})$.

One route, employed in the present paper, for avoiding the problem at least on the cohomological level is to construct a V with $c_2(X) - c_2(V) = 0$. Alternatively the problem could be circumvented if one could solve the anomaly constraint with the help of a second bundle in the hidden sector which realizes the class W as $c_2(V_{\mathrm{hid}})$ (cf. also [19]), leaving $V = V_{\mathrm{obs}}$ intact. Thereby one stays within the framework of perturbative $(0, 2)$ models characterized by stable holomorphic bundles (V, V_{hid}) of $c_1(V) = c_1(V_{\mathrm{hid}}) = 0$ (this condition can be relaxed) with

$$c_2(V) + c_2(V_{\mathrm{hid}}) = c_2(X) \tag{1.5}$$

(cf. in this connection also [30]). This leads therefore to the following general problem:

- Suppose an effective holomorphic curve class (a sum of irreducible holomorphic curves with non-negative integral coefficients) is given[1] which represents the compact support of the (space–time filling) fivebrane and whose cohomology class is denoted by

$$W \in H^4(X, \mathbf{Z}). \tag{1.6}$$

When can W be represented as the second Chern class of a vector bundle (of $c_1(V_{\text{hid}}) = 0$)

$$W = c_2(V_{\text{hid}})? \tag{1.7}$$

Here V_{hid} and V have to be stable with respect to the same Kähler class.

Although a precise, or at least sufficient condition is not known, one knows a necessary condition: a holomorphic vector bundle V_{hid} (stable w.r.t. J) satisfies the Bogomolov inequality

$$0 \leq c_2(V_{\text{hid}})J. \tag{1.8}$$

The ensuing necessary condition $0 \leq WJ$ is satisfied in our case as W was an effective class.

In [25] the conjecture is put forward that a stable bundle (more precisely reflexive sheaf) V_{hid} of rank r_{hid} and[2] $c_1(V_{\text{hid}}) = 0$ exists if one has for some ample class D that

$$c_2(V_{\text{hid}}) - \frac{r_{\text{hid}}}{24} c_2(X) = r_{\text{hid}} D^2. \tag{1.9}$$

Assuming this conjecture to be true (1.7) can be solved if (1.9) holds for W. Proving (1.7) this way, one needs not only W effective with $0 \leq WJ \leq c_2(X)J$ (the latter from $0 \leq c_2(V)J$), but even the following condition (note $0 \leq c_2(X)J$ by the Miyaoka–Yau theorem):

$$\frac{r_{\text{hid}}}{24} c_2(X)J \overset{!}{\leq} W\,J \leq c_2(X)J. \tag{1.10}$$

1.2. *Problems when no fivebranes are present*

Avoiding fivebrane contributions is of course not the end of the problems for consistent heterotic string compactifications caused by the anomaly cancellation requirement. For this let us assume that no need arises for a current representing a fivebrane contribution (something which is already cohomologically detectable); or assume that one has succeeded in representing such a contribution by a bundle in the hidden sector. Then still one has the problem that (1.2) actually has to be solved locally, i.e., *on the form level*. If one is not in the quite exceptional case of having an $F \neq R$ with $\text{tr}\, F \wedge F = \text{tr}\, R \wedge R$ locally, this will necessitate to turn on of a non-trivial H-field (though this time smooth and not being singular to balance by its dH a delta-function contribution δ_W).

However, having now a non-trivial H-field turned on, the compatibility conditions (stemming from the requirement of supersymmetry) concerning the geometry of the underlying compactification space and the H-field configuration, demand that X is non-Kähler [1]; cf. also [21–23]. The severe consistency problem stems from the fact that the non-trivial H-field (induced from a mismatch between $\text{tr}\, R \wedge R$ and $\text{tr}\, F \wedge F$) has actually to be used at the same time consistently on the right hand side of the anomaly balance for the connection $\omega + aH$ from which $\text{tr}\, R \wedge R$ is computed. Here ω is the torsion-free spin connection and H understood as a 1-form by suitable contractions with vielbeins. On the level of the 3-form H itself this amounts to a cubic relation $H = dB + \alpha' (\Omega_3(\omega + aH) - \Omega_3(A))$ with the corresponding Chern–Simons terms (where $\Omega_3(A) = \text{tr}(A \wedge F - \frac{1}{3} A \wedge A \wedge A)$).

Here we are in the case that the Hodge type of the H field is $(2, 1) + c.c.$ as dH has to be of type $(2, 2)$. (More generally one may consider also the case of a Hodge type $(3, 0) + c.c.$ which could be cancelled in the complete square part of the Lagrangian by a non-trivial gaugino condensate vev [21,23]). This has the consequence that the DUY theorem, assuring for stable bundles the solvability of the equations of motion from a topological condition, is no longer applicable. Here one has to note that for X being non-Kähler already the notion of stability is slightly

[1] Strictly speaking the assumption on W is more specific as it is $c_2(X) - c_2(V)$ for a stable bundle V.

[2] In addition $c_3(V_{\text{hid}})$ can be conjecturally chosen freely if it is $< \frac{16\sqrt{2}}{3} r_{\text{hid}} D^3$.

modified as the would-be Kähler form J is no longer closed with corresponding consequences for $\int_X c_1(V)J^2$ and the notion of slope [2,29]. For steps in the direction of a generalization, cf. [26] for the case of a T^2-fibration with base B a $K3$ surface, which from the normal ($H = 0$) perspective would be a rather degenerate exceptional case.

Corresponding problems in the strongly coupled heterotic set-up

The need to solve the anomaly constraint actually locally and not only in the global (cohomological) balance becomes especially palpable in the case of heterotic M-theory where

$$(dG)_{IJKL\,11} \sim \kappa^{\frac{2}{3}} \left(\sum_{i=0}^{1} \delta(x_{11} - x_i) \left(\frac{1}{2} \mathrm{tr}\, R_i \wedge R_i - \mathrm{tr}\, F_i \wedge F_i \right) + 16\pi^2 \sum_{k=1}^{m} \delta(x_{11} - x_k) \delta_{W_{x_k}} \right) \wedge dx_{11}.$$

The (space–time filling) fivebrane contribution W consists of m components W_{x_k}, supported (compactly) on various (itself not necessarily irreducible) holomorphic curves C_k (of dual 4-forms δ_{W_k} in X_k) lying in the Calabi–Yau space X_k over the point x_k of the interval $I = [x_0, x_1]$. Here the standard embedding $F_{\mathrm{obs}} = R$, "spin in the gauge", can no longer be a solution, although still fulfilling the global balance. The reason is that the local character, here along I, of the condition becomes especially pronounced, demanding that at each end of the interval one of the individual E_8 bundles V_i cancels the term $\frac{1}{2}\mathrm{tr}\; R_i \wedge R_i$. The idea of representing W as $c_2(V_{\mathrm{hid}})$ becomes here obsolete as this strategy is not local in I, let alone local in X. (For some solutions with $W = 0$ locally in I, though not in X, cf. [27].)

Now, considered locally in I, it seems that fivebrane components W_{x_k} can easily be balanced consistently by a corresponding $G_{2,2;0}$-contribution[3] of the form step function $\theta(x_{11} - x_k)$ times the $(2, 2)$ form $\delta_{W_{x_k}}$. Similarly, to absorb a mismatch $\frac{1}{2}\mathrm{tr}\, R_i \wedge R_i - \mathrm{tr}\; F_i \wedge F_i$ one can also use a boundary contribution $G = \theta$ times $(2, 2)$ which therefore, it seems, considered just locally in I, could absorb both types of contribution from the right hand side.

However, actually the various contributions, locally constant along x_{11}, have to fit together in the "upstairs picture" on $\mathbf{S}^1$ with a $\mathbf{Z}_2$ action, i.e., the various jumps in total have to compensate each other. As G is odd, jumps in the bulk cancel mutually and the fixpoint contributions remain. So, having no H-type boundary components in G, one ends up with

$$0 = \sum_{i=0}^{1} \left(\frac{1}{2} \mathrm{tr}\, R_i \wedge R_i - \mathrm{tr}\; F_i \wedge F_i \right) + 16\pi^2 \sum_{x_k = x_0 \text{ or } x_1} \delta_{W_{x_k}}. \tag{1.11}$$

As the smooth and delta-function parts have to cancel individually, one gets (1.11) just for the smooth terms and no fivebranes on the boundaries (anti-fivebranes being forbidden).

1.3. Discussion

Let us emphasize that $W = 0$, seemingly avoiding $G \neq 0$, is satisfied in the aforementioned models [27] only at the cohomological level. Realizing that the condition to be solved is on the differential form level, one does not get $G = 0$ models. Similar remarks apply to older $(0, 2)$ models which solve the anomaly without fivebranes but only on the cohomological level, and to the models we present in this paper. Nevertheless let us point out two things. The ability to avoid the delta-function contribution already reduces substantially the problem of potential inconsistencies when one tries to solve the anomaly constraint (1.2) as the problem with singularities occurring on both sides in different orders is avoided. The general expectation is then that some relevant features of the models (understood in the naive $H = 0$ sense) persist, despite the remaining necessity to adjust, perhaps order by order, a non-trivial (but now smooth) H-field configuration. But note the caveat that the radius will be fixed to a finite value, so there will be no large radius limit for a perturbative supergravity treatment; also a stability notion for X non-Kähler is more subtle [29]. The same philosophy underlies procedures for checking that the number of fivebranes wrapped on elliptic fibers in heterotic spectral cover models matches the number of threebranes in a dual F-theory model [3,31]; or also the numerous phenomenological studies of heterotic compactifications done so far. The other point, specific to our

[3] Boundary $G_{2,1;1}$-components of type $H^{(2,1)}(+c.c.)$ times $\delta(x_{11} - x_0) \wedge dx_{11}$, if possible, would bring one back to the previous problems, so we assume these to be absent (also with such a contribution the connection $\omega + a\delta(x_{11} - x_i)H$ used in $1/2\,\mathrm{tr}\, R_i \wedge R_i$ would have a problematic delta-function singularity). So a potential consistency problem from the connection $\omega + aH$ is not induced as $H = 0$. Still the volume modulus can be stabilized (without $H^{(2,1)} \neq 0$, X non-Kähler) by worldsheet instantons [23] or the S-Track mechanism [24].

$W = 0$ models, is that they live still on elliptically fibered Calabi–Yau spaces. For many investigations touching more conceptual questions (dualities with other string models is a prominent example) these seemingly more abstract compactifications have turned out to be more suitable than Calabi–Yau spaces given by embeddings in a weighted projective space (or products of them).

In Section 2 we discuss the various options for the (bases B of the elliptically fibered) spaces over which we build our bundles. In Section 3 we recall some notions related to stability and describe in Section 4 the way we build our bundles which enable us to get $W = 0$ (this contrasts with the fact [3] that bundles built with the spectral cover method give $W \neq 0$). There the stability of the bundles could be established more directly [10] whereas here the issue of stability becomes a major technical point to which we do full justice only in the mathematical companion paper [28]. In Section 5 we give for B an Enriques surface the relevant stability results and show that within our class of bundles $W \geq 0$ is violated. Stability results and $W = 0$ examples for other bases are given in Section 6.

2. The elliptic Calabi–Yau space over various bases

As our procedure to build bundles on X involves an extension of bundles $U_p = \pi^* E_p$ which are themselves pull-backs of bundles E_p on the base B, one has to ensure the stability of such pull-back bundles U_p if E_p is stable. For this we distinguish two cases among our set of bases, consisting of Hirzebruch surfaces $\mathbf{F}_r$ ($r = 0, 1, 2$), del Pezzo surfaces dP_k ($k = 0, \ldots, 8$) and the Enriques surface E. The search for a specific gauge group, be it of a GUT theory or of the Standard Model, motivates the choice of a Hirzebruch surface or the Enriques surface as base. On the one hand we will treat the Enriques surface E which is also of special physical importance as a GUT group $SU(5)$ can be broken to the Standard Model group because of $\pi_1(X_E) = \mathbf{Z}_2$. On the other hand we will treat in Section 6 the case where the anticanonical bundle is ample, i.e. the remaining cases except $\mathbf{F}_2$.

2.1. The case of B an Enriques surface

Let us describe the different physical and mathematical issues related to the use of the Enriques surface as base.

2.1.1. On the physical motivation for the Enriques surface

One approach to realizing the Standard Model gauge group within heterotic string compactifications is to build a bundle V of structure group $SU(5)$, leading to an unbroken gauge group given by the grand unified group $SU(5)$ in the observable sector. In that case $\pi_1(X) = \mathbf{Z}_2$ allows for a Wilson line, breaking the commutator $SU(5)$ (in E_8) of a structure group $SU(5)$ to the Standard Model gauge group. For this one needs a Calabi–Yau space whose non-trivial fundamental group contains a $\mathbf{Z}_2$. X is non-simply connected only if the base B of the fibration is given by an Enriques surface E where $\pi_1(X) = \pi_1(B) = \mathbf{Z}_2$ (where $c_1 := c_1(B)$ is a two-torsion class).

This approach is in contrast to the procedure where one starts from a simply connected Calabi–Yau threefold X' having a free involution τ from which the required non-simply connected Calabi–Yau space X is built as $X = X'/\mathbf{Z}_2$. The existence of τ is related to the existence of a second section of the elliptic fibration [11–18]. Then one searches for τ-invariant bundles having six generations.

By contrast, in the case of the Enriques Calabi–Yau, one searches directly on X for bundles of net generation number $N_{\text{gen}} = \pm 3$. This, however, led for spectral bundles to the following problem. From the mismatch of anomaly cancellation between $c_2(V)$ and $c_2(X)$ one has to introduce a number of fivebranes of total cohomology class $W = w_B \sigma + a_f F$. One has the effectivity condition $w_B = 12c_1 - \eta = -\eta \geq 0$ where $\eta \in H^2(B, \mathbf{Z})$ is a datum describing the bundle (the spectral surface C has cohomology class $n\sigma + \eta$ where n is the rank of V). η must be an effective class satisfying $\eta \geq nc_1$ which for n even reduces to $\eta \geq 0$ and for n odd to $\eta \geq c_1$; so $\eta = 0$ or c_1, giving $N_{\text{gen}} = \lambda\eta(\eta - nc_1) = 0$. (For another attempt cf. [19].)

2.1.2. Mathematical details on the Enriques surface

Consider standard (fibre type A) elliptically fibered CY spaces X with one section [3] and base given by an Enriques surface [20], i.e., $h^{1,0}(B) = 0$ and $K_B^2 = \mathcal{O}_B$. B has non-trivial Hodge numbers $h^{1,1} = 10$, $h^{0,0} = h^{2,2} = 1$, so $c_1^2 = 0$ and $c_2 = 12$, and middle cohomology

$$H^2(B, \mathbf{Z}) = \mathbf{Z}^{10} \oplus \mathbf{Z}_2 \quad \text{with intersection lattice } \Gamma^{1,1} \oplus E_8^{(-)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8^{(-)} \tag{2.1}$$

(orthogonal decompositions). Further $\phi c_1 = 0$ for all $\phi \in H^2(B, \mathbf{Z})$. B is always elliptically fibered with fibre f over $b = \mathbf{P}^1$. However two of the fibers, f_1 and f_2, are double fibers: $f = 2f_i$, which prevents B from having a section and $c_1 = f_1 - f_2$ is not effective.

On a generic ('unnodal') B no smooth rational curves exist and all irreducible curves C have $C^2 \geq 0$. The integral classes in one of the two components of the cone in $H^2(B, \mathbf{R})$ defined by $C^2 \geq 0$ constitute the effective cone (adding the torsion class c_1 does not matter for this if $C \neq 0$). For C nef (i.e., $DC \geq 0$ for all curves C on B) $|C|$ is base-point-free, C is ample if also $C^2 \geq 6$ [20]. $C = xa + yf = (x, y) \in \Gamma^{1,1}$ is nef for $x, y \geq 0$.

B can be represented as the quotient of a $K3$ surface by a free involution. The corresponding $\pi_1(B) = \mathbf{Z}_2$ is inherited by the elliptic Calabi–Yau space X which itself is a quotient by a free involution on $K3 \times T^2$ (also acting as $z \to -z$ on the T^2). The holomorphic 2-form Ω_2 of $K3$ being odd, the holomorphic 3-form $\Omega_2 \wedge dz$ is preserved, the quotient X being a Calabi–Yau space of vanishing Euler number.

3. The condition of stability

We will choose as polarization $J = z\sigma + \pi^* H$ where H (chosen in the integral cohomology) is in the Kähler cone $\mathcal{C}_B$ of the base B and $z \in \mathbf{R}^{>0}$. For an elliptically fibered Calabi–Yau space X one has that J is a Kähler class if [18]

$$J \in \mathcal{C}_X \Longleftrightarrow z > 0, \qquad H - zc_1 \in \mathcal{C}_B. \tag{3.1}$$

Stability of a bundle V (with respect to J) means $\mu_J(V') < \mu(V)$ for all coherent subsheaves V' of V of $rk\, V' \neq 0, rk\, V$. Here $\mu(V) = \frac{1}{rk\, V} \int c_1(V) J^2$ is the slope of V with respect to J. Similarly semistability is defined by the condition $\mu_J(V') \leq \mu(V)$.

A bundle V stable w.r.t. J satisfies the Bogomolov inequality

$$c_2(V) J \geq 0. \tag{3.2}$$

The decomposition of the fivebrane class $W = w_B \sigma + a_f F = c_2(X) - c_2(V)$ becomes for B an Enriques surface $12F - c_2(V)$ which gives then

$$c_2(V) J = -w_B H + z(12 - a_f). \tag{3.3}$$

For bundles built by the spectral cover construction one knows by [10], Thm. 7.1, that suitable J must have z sufficiently small (*"spectral polarizations"*). But if z has to be chosen negligible small, only $w_B = 0$ is possible as $-w_B H \leq 0$ by the requirement $w_B \geq 0$. The latter stems from the condition that the fivebrane class is effective. On the other hand, below we will assure stability for z sufficiently *large*; note that on B the Enriques surface there will be no problem then with the criterion (3.1) when just choosing $H \in \mathcal{C}_B$.

Let us consider now the stability of zero-slope bundles V constructed as extensions

$$0 \to U \to V \to W \to 0. \tag{3.4}$$

Here U and W are assumed to be stable. Necessary conditions for the stability of V are that

- $\mu(U) < 0$
- the W of $\mu(W) > 0$ is not a subbundle of V, i.e., the extension (3.4) is non-split.

4. Bundles built as extensions

As mentioned earlier, one problem in heterotic model building, especially on elliptically fibered Calabi–Yau spaces X, is the occurrence of a number of space–time filling fivebranes [3], preventing the model from being interpreted as a perturbative non-linear sigma model. More specifically, this problem occurs within the spectral cover construction [3,4] (equivalently understood as a relative Fourier–Mukai transform [7]). The advantage of this method is an improved flexibility in the explicitly computed net generation number [5,6].

This problem is circumvented in the present paper as follows. We define a stable $SU(n)$ bundle V as a non-trivial extension of bundles U and W of lower rank, especially when W is a line bundle. The question of stability, and already the existence of a non-split extension, turns out to be non-trivial. The bundles U and W are constructed as pull-backs from the base B, twisted by certain line bundles. For B a Hirzebruch surface $\mathbf{F_r}$ we find GUT models with chiral matter, and for B the Enriques surface the Standard Model gauge group. In the GUT case it is possible to avoid fivebranes in the anomaly constraint. Thereby one stays within the framework of perturbative (0, 2) models, characterized by stable holomorphic bundles V of $c_1(V) = 0$ (this condition can be relaxed) satisfying

$$c_2(V) = c_2(TX). \tag{4.1}$$

Now, in the spectral cover construction the bundle decomposes on the generic fibre F as

$$V|_F = \bigoplus_{i=0}^{n} \mathcal{O}_F(p_i - p_0) \quad \text{with} \sum p_i = p_0 \tag{4.2}$$

and is adiabatically extended along the base: the system of the (p_i) becomes base-point dependent, leading to an n-fold cover C of B, including a twist by a line bundle $\mathcal{O}_B(\eta)$ on B.

A different starting point is to choose V on F as

$$V|_F = \bigoplus \mathcal{O}_F(x_i p_0) \quad \text{with} \sum x_i = 0. \tag{4.3}$$

One of the simplest possibilities is to choose $x_i =: -x$ for $i = 1, \ldots, n$ and $x_0 = nx$. Thus, rewriting (4.3), one starts from a *split* short exact sequence

$$0 \longrightarrow \mathcal{O}_F^n \otimes \mathcal{O}_F(-xp_0) \longrightarrow V|_F \longrightarrow \mathcal{O}_F(nxp_0) \longrightarrow 0. \tag{4.4}$$

To spread this out along B one chooses now V_{n+1} as the extension

$$0 \longrightarrow \pi^* E_n \otimes \mathcal{O}_X\left(-x\sigma - \pi^*\alpha\right) \longrightarrow V_{n+1} \longrightarrow \mathcal{O}_X\left(n(x\sigma + \pi^*\alpha)\right) \longrightarrow 0. \tag{4.5}$$

Here the non-trivial information about the bundle along the direction B is encoded in E_n, including a twist by a line bundle $\mathcal{O}_B(\alpha)$ of the base. More generally, one may consider the case where the set of the x_i partitions into two sets of $x_i = px$ for qi's and $x_i = -qx$ for pi's, which globally corresponds to the extension (with $U_p = \pi^* E_p$, $W_q = \pi^* E_q$ and $D = x\sigma + \pi^*\alpha$)

$$0 \longrightarrow U_p \otimes \mathcal{O}_X(-qD) \longrightarrow V_{p+q} \longrightarrow W_q \otimes \mathcal{O}_X(pD) \longrightarrow 0 \tag{4.6}$$

(where one demands $DJ^2 > 0$ so that the slope condition $\mu(U_p \otimes \mathcal{O}(-qD)) < 0$ is fulfilled). So the class of bundles we consider is given by bundles V_{p+q} of rank $p + q$ defined as non-trivial extensions (4.6) of stable bundles U_p and W_q of rank p and q with $c_1(U_p) = 0 = c_1(W_q)$, suitably twisted by powers of a line bundle $\mathcal{O}(D)$ so as to preserve $c_1(V_{p+q}) = 0$.

One can consider in particular the case that U_p and W_q are pull-backs $\pi^* E_p, \pi^* E_q$ of bundles on B. Then $\pi^* E$ is (semi)stable if E is (semi)stable on B with respect to H, say for B the Enriques surface; the detailed arguments for this and all mathematical statements concerning the stability proofs and non-split conditions below are given in the mathematical companion paper [28]. Later we will actually show stability of V_{p+q} only for $q = 1$.

One finds with $c_2(U_p) = uF$ and $c_2(W_q) = wF$

$$c_2(V_{p+q}) = -\frac{1}{2} pq(p+q)D^2 + (u+w)F. \tag{4.7}$$

Let us now come to the physical conditions. One must ensure the effectivity of the class

$$W = w_B \sigma + a_f F = c_2(X) - c_2(V_1) - c_2(V_2) \tag{4.8}$$

of the fivebrane. Assuming for simplicity no hidden sector bundle one finds as components

$$w_B = 12c_1 + \frac{1}{2} pq(p+q)x(2\alpha - xc_1), \qquad a_f = c_2 + 11c_1^2 + \frac{1}{2} pq(p+q)\alpha^2 - (u+w). \tag{4.9}$$

For the net chiral matter content one finds as generation number

$$N_{\text{gen}} = \frac{1}{2} c_3(V_{p+q}) = \frac{pq}{6}(p^2 - q^2)D^3 + x(qu - pw) + \frac{1}{2} c_3(U_p) + \frac{1}{2} c_3(W_q). \tag{4.10}$$

Note that in the special case that U_p and W_q are pull-backs from the base (so that they have vanishing third Chern class) one finds $N_{\text{gen}} \sim x$ over the Enriques base.

Below we will prove stability of V_{p+q} for $q = 1$ given a stable bundle U_p. To get a concrete stable bundle U_p we take $U_p = \pi^* E_p$. Over B with K_B^{-1} ample one gets then GUT models with $W = 0$ and $N_{\text{gen}} \neq 0$ (cf. Section 6). Over the Enriques base one can get the Standard Model gauge group; then, however, one encounters the side effect that N_{gen} will run just with x as w_B does; furthermore, only the case $w_B = 0$ would be allowed (as will be seen below), giving $x = 0$; for $x = 0$ stability, however, cannot be assured.

5. Standard Model groups: Enriques base

For the case $(p, q) = (n, 1)$, V_{n+1} can be shown to be stable. So let V_n be a stable bundle of $c_1(V_n) = 0$ and $D = x\sigma + \pi^*\alpha$ and define V_{n+1} as a non-split extension (here $W_1 = \mathcal{O}$)

$$0 \to V_n \otimes \mathcal{O}(-D) \to V_{n+1} \to \mathcal{O}(nD) \to 0. \tag{5.1}$$

Let us first discuss the non-split condition. For this assume $V_n = \pi^* E_n$ with $c_1(E_n) = 0$ and E_n stable. Then one finds for $x > 0$ and for $a := \alpha H < 0$ the index condition [28]

$$I := n - c_2(E) + \frac{n(n+1)^2}{2}\alpha^2 > 0 \tag{5.2}$$

for precisely when a non-split extension exists. If $x \leq 0$ then $Ext^1 \neq 0$ exactly if $I < 0$.

Stability of V_{n+1} *and physical constraints*

We note the following necessary condition: if V_{n+1} is stable (so (5.1) is non-split) then

$$x \neq 0 \Longrightarrow x \cdot a < 0 \tag{5.3}$$

where $a := \alpha H$. V_{n+1} has now specific stability regimes w.r.t. the Kähler class $J = z\sigma + \pi^* H$

$$0 < x < -a \longrightarrow \frac{nx}{-na+1}\frac{H^2}{2} < z < \frac{nx}{-na}\frac{H^2}{2} \tag{5.4}$$

$$-a < x < 0 \longrightarrow \frac{nx}{-na}\frac{H^2}{2} < z < \frac{nx}{-na+1}\frac{H^2}{2}. \tag{5.5}$$

The physical constraints concern the effectivity of the fivebrane $W = w_B\sigma + a_f F$ where

$$w_B = \frac{1}{2}n(n+1)x(2\alpha - xc_1) \geq 0, \qquad a_f = 12 + \frac{1}{2}n(n+1)\alpha^2 - c_2(E) \geq 0 \tag{5.6}$$

and the phenomenological value ± 3 of the net generation number

$$N_{\text{gen}} = x\left(\frac{1}{2}n(n^2-1)\alpha^2 + c_2(E)\right). \tag{5.7}$$

Note that, as $w_B \geq 0$ requires therefore $x\alpha \geq 0$, one gets in view of (5.3)

$$w_B \geq 0 \Longrightarrow x = 0. \tag{5.8}$$

If a hidden sector bundle of the same type is turned on, the argument remains valid as $w_B = \sum_{i=1}^2 c_i\, x_i \alpha_i \geq 0$ (with $c_i > 0$, $c_1 = \frac{n(n+1)}{2}$) gives $w_B H \geq 0$, a contradiction to (5.3).

$x = 0$ is the case for which the existence of stable bundles could not be assured above.

6. GUT groups: Working over B with ample K_B^{-1}

In this section B denotes a surface with ample K_B^{-1}. Now $\pi^* E$ is (semi)stable on X with respect to $J = z\sigma + \pi^* H \in \mathcal{C}_X$ if E is (semi)stable on B with respect to $H = hc_1$; here $H - zc_1 \in \mathcal{C}_B$ gives $z < h$. Given the fact that c_1 is now no longer a two-torsion class, a greater numerical freedom between W and $N_{\rm gen}$ occurs; in particular the common proportionality to the parameter x is lifted. Therefore it is possible here to have $W = 0$ and $N_{\rm gen} \neq 0$.

Concretely the fivebrane class has components (assuming for simplicity $V_{\rm hid} = 0$)

$$w_B = 12c_1 + \frac{1}{2}n(n+1)x(2\alpha - xc_1), \qquad a_f = c_2 + \frac{1}{2}n(n+1)\alpha^2 - c_2(E) + 11c_1^2. \tag{6.1}$$

In contrast to the case of the Enriques base it is now possible to satisfy $w_B \geq 0$ while having $x \neq 0$. One finds now $W = 0$ (just to get $w_B, a_f \geq 0$ is easy) for the choices

$$\alpha = \left(\frac{x^2}{2} - \frac{12}{n(n+1)}\right)\frac{c_1}{x} \Longrightarrow w_B = 0 \tag{6.2}$$

$$c_2(E) = c_2 + 11c_1^2 + \frac{n(n+1)}{2}\alpha^2 \Longrightarrow a_f = 0. \tag{6.3}$$

For instance, for building an $SO(10)$ GUT model without fivebranes one can use the twist $D = \sigma - \pi^* c_1/2$ and a rank $n = 3$ bundle E on a base $\mathbf{F_r}$ of instanton number 104. Or one may construct an E_6 GUT model without fivebranes from using the twist $D = 2\sigma$ and a plane bundle of $c_2(E) = 92$. (One immediately checks the non-split conditions.)

One gets furthermore that

$$N_{\rm gen} = x\left(\frac{n(n^2-1)}{6}\left(3\alpha^2 - 3x\alpha c_1 + x^2c_1^2\right) + c_2(E)\right). \tag{6.4}$$

Let us mention that one can carry through a similar program also for extensions by spectral bundles (without special restrictions on the base surface B), leading to examples of stable bundles without fivebranes; cf. [28].

So the general lesson in all the different cases is similar: the greater numerical freedom provided by the twist and the extension can allow one to have $W = 0$, the burden then is however to prove stability of the extensions [28].

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